e-Coli
And other problems

Paul Scott
Adelaide, SA
<mail@paulscott.info>

In applied mathematics particularly, we are interested in modelling real life situations; that is, we try to express some actual phenomenon mathematically, and then use our mathematics to determine future outcomes. It may be that we actually wish to change the future outcome. Our mathematics will not do this, but at least it tells us what to expect.

Populations

Let us make the following assumptions about rabbits:

(i) rabbits live forever(!)
(ii) every month each pair begets a new pair.

1. Complete the table below which shows the growth of the rabbit population beginning with a single pair.

Number of months	0	1	2	3	4	5
Total population (in pairs)	1	2				

Illustrate your results with a step-graph.

(At first sight you might have expected to obtain a Fibonacci sequence here. What went wrong?)

Of course the rabbits would have to be very well organised for their population to increase in sudden jumps like this. For a large population it is more realistic to think of the increase occurring continuously. We can approximate this behaviour by considering a very large number of small jumps, as illustrated in the following example.

In 2002, the world population P_0 was 6 billion, i.e., 6×10^9. Allowing for deaths, the growth rate is (approximately) 2% per year. What was the population in 2008?

It is easy to see that this is just a compound interest problem in disguise, and at a very poor rate of interest at that!

(i) If the population increases in annual jumps, then in year t the population with be $P_t = P_0(1 + 0.02)^t$.

(ii) If the population increases in half-yearly jumps, then by year t there will have been $2t$ jumps of $1/2 \times 2\%$ so the population will be
$$P_t = P_0\left(1+\frac{0.02}{2}\right)^{2t}$$

(n) If the population increases in jumps n times a year, then by year t there will have been nt jumps of $1/n \times 2\%$ so the population will be
$$P_t = P_0\left(1+\frac{0.02}{n}\right)^{nt}$$

Now what can we say about P_t as n gets large? If we write $n = 0.02m$, then
$$\begin{aligned} P_t &= P_0\left(1+\frac{0.02}{n}\right)^{nt} \\ &= P_0\left(1+\frac{1}{m}\right)^{0.02m.t} \\ &= P_0\left(1+\frac{1}{m}\right)^{m\times 0.02t} \end{aligned}$$

Now as n gets large, so does m, and as we have seen in the previous *AMT* (vol. 65 no. 3),
$$\left(1+\frac{1}{m}\right)^{m}$$
approaches e. Hence for large n, $P_t \approx P_0e^{0.02t}$, and for continuous growth, taking the limit, $P_t = P_0e^{0.02t}$.

Thus in 2008, since we are counting from 2002, $t = 6$, $P_0 = 6 \times 10^9$, and the new population is given by

$$\begin{aligned} P_5 &= (6 \times 10^9)e^{0.02 \times 6} \\ &= (6 \times 10^9)e^{0.12} \\ &= (6 \times 10^9)1.128 \\ &= 6.786 \times 10^9 \end{aligned}$$

using a calculator to calculate to calculate e raised to the power 0.12.

2. Using the formula $P_t = P_0e^{0.02t}$, find the year in which the 2002 world population will double. (Note that $e^{0.7} \approx 2$.)

For a population increasing continuously at a rate of r% per year, the population in year t is given by $P_t = P_0e^{(r/100)t}$ where P_0 is the initial population.

Bacteria

The bacteria known as *E. coli* live quite happily in our bodies — for the most part. Occasionally a rare strain develops causing an infection with very unpleasant consequences. One of the problems is that bacteria can multiply at a very fast rate — so we are back to the population problem again.

3. The number N_t of bacteria in a culture t seconds after it is established is $N_t = 1000e^t$. How many bacteria are present initially? How many are present after (a) 1 second? (b) 2 seconds? (c) 3 seconds?

This is a pretty rapid rate of increase!

Food shortages

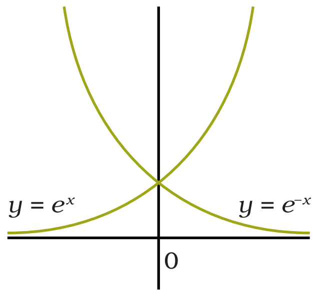
4. It has been calculated that one ninth of a hectare is required to provide food for one person. The world contains 4×10^9 hectares of arable land. Hence the world population would seem to be limited to 36 billion. If the population continues to grow at the rate of 2% per year, when will it reach this figure?

Assume that the population in 2002 was 6 billion. You will need to know (calculate!) that $e^{1.8} \approx 6$.

Radiation

In 1898, Marie Curie discovered radioactive radium which was then developed for useful medical purposes. However, since the 1945 bombing of Hiroshima, the world has lived under some threat of nuclear annihilation — and the problem with nuclear material is that, for better or worse, we are stuck with it!

A radioactive substance decays at a rate directly proportional to the amount remaining. In fact, the half life of a radioactive substance is the time it takes for half the original amount to decay. It turns out that our exponentials, which we have so far associated with rapid increase, also occur with very slow decrease. We achieve this by using the function $y = e^{-x}$ in place of $y = e^x$ (see the graphs below).



5. After t years, the number of grams N_t of radium left out of an original ten grams is given by $N_t = 10e^{-0.0004t}$.
   (a) How many grams are left after 1000 years?
   (b) What is the half-life of radium?
   You might like to use $e^{-0.4} \approx 0.67$, $e^{-0.65} \approx 0.5$.

This fairly typical long half-life illustrates why many scientists are reluctant to lock future generations into storing large amounts of nuclear waste.

Bibliography

Scott, P. R. (1974). *Discovering the mysterious numbers*. Cheshire.

Scott, P. R. (n.d). *The powers that be*. Accessed at http://web.me.com/paulscott.info/xcal/61/61.html.